NORTH-HOLLAND

On Square Roots of *M*-Operators

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ABSTRACT

Using a cone order in a real Banach space, the concept of an M-operator is discussed, and the existence and uniqueness of the square roots of M-operators are studied. In this way most of the results of the paper by Alefeld and Schneider are generalized to an infinite dimensional case. For some finite dimensional situations our results also seem to be new.

1. INTRODUCTION

This contribution is a broadening of the square root results obtained in [1] for the case of the "classical" M-matrices. Our generalization is concerned with a class of M-operators on a rather general partially ordered real Banach space $\mathscr{E}$ generated by a closed normal cone $\mathscr{K}$. It is interesting that most of the results obtained in [1] for $n \times n$ matrices can be generalized with very minor changes of the proofs for operators in partially ordered infinite dimensional spaces. Some of these changes are simply "enforced" because of lack of particular tools typical for finite dimensional case in the infinite dimensional situation. Mostly, however, a basis independent operator approach sheds much light on the interplay of various aspects of the problem. In this connection, let us mention e.g. an explanation of the nonuniqueness

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result contained in Example 1 in Section 3. It is interesting even for square roots of $n \times n$ matrices. Also, the cone $\mathscr{K}$ introducing a partial order into $\mathscr{E}$ is general in the sense that it may have its interior Int $\mathscr{K}$ empty and it need not be a lattice type cone, i.e., $x, y \in \mathscr{K}$ need not imply that $\sup\{x, y\}$ either exists or belongs to $\mathscr{K}$.

2. DEFINITIONS, NOTATION, AND PRELIMINARIES

Let $\mathscr{E}$ be a real Banach space, $\mathscr{E}'$ its dual, and $B(\mathscr{E})$ the space of bounded linear operators on $\mathscr{E}$. It is assumed that $\mathscr{E}'$ and $B(\mathscr{E})$ are equipped by the usual norms and thus they are Banach spaces as well. Let $\mathscr{F}$ denote the *complex extension* of $\mathscr{E}$, i.e. $\mathscr{F} = \mathscr{E} \oplus i\mathscr{E}$, with the norm

$$\|z\|_{\mathscr{F}} = \sup\{\|x \cos \theta + y \sin \theta\|_{\mathscr{E}} : 0 \leqslant \theta \leqslant 2\pi\},$$

where $z = x + iy$, $x, y \in \mathscr{E}$.

Let $K \subset \mathscr{E}$ be a closed normal and generating cone [7], i.e., $\mathscr{K}$ satisfies the following relations:

(i) $\mathscr{K} + \mathscr{K} \subset \mathscr{K}$,
(ii) $a\mathscr{K} \subset \mathscr{K}$,
(iii) $\mathscr{K} \cap (-K) = \{0\}$,
(iv) $\bar{\mathscr{K}} = \mathscr{K}$ ($\bar{\mathscr{K}}$ denotes the norm closure of $\mathscr{K}$),
(v) $\mathscr{E} = \mathscr{K} - \mathscr{K}$, i.e., for every $y \in \mathscr{E}$, there exist $y_j \in \mathscr{K}$, $j = 1, 2$, such that $y = y_1 - y_2$, and
(vi) There exists a real $\delta > 0$ such that $\|x + y\|_{\mathscr{E}} \geq \delta\|x\|_{\mathscr{E}}$ whenever $x, y \in \mathscr{K}$.

Let $\mathscr{K}'$ be the *dual cone*, i.e.

$$\mathscr{K}' = \{x' \in \mathscr{E}' : \forall x \in \mathscr{K}, [x, x'] \equiv x'(x) \geqslant 0\},$$

and $\mathscr{K}^d$ the *d-interior* defined as

$$\mathscr{K}^d = \{x \in \mathscr{K} : \forall x' \in \mathscr{K}', x' \neq 0 \Rightarrow [x, x'] > 0\}.$$

In this paper a stronger demand concerning the closedness of the cone $\mathscr{K}$ is needed:

(vii) $\mathscr{K}$ is weakly closed.

Obviously, if $\operatorname{Int}\mathscr{K} \neq \varnothing$, then $\operatorname{Int}\mathscr{K} = \mathscr{K}^d$. A closed normal cone $\mathscr{K}$ having $\operatorname{Int}\mathscr{K} \neq \varnothing$ is called *solid*.

A partial order is introduced into $\mathscr{E}$ by setting

$$x \leqslant y \ \text{(or equivalently } y \geqslant x) \quad \Leftrightarrow \quad (y - x) \in \mathscr{K}.$$

A partially ordered space E generated by a cone K is called a *Dedekind type* space if every nondecreasing bounded sequence is norm convergent, i.e., the following implication holds: $\forall \{x_k\} : x_k \in \mathscr{K},\ x_k \leqslant x_{k+1} \leqslant x$, where x is independent of k, implies the existence of an element $x^* \in \mathscr{K}$ such that $\lim_{k\to\infty} \|x_k - x^*\|_{\mathscr{E}} = 0$.

REMARK 1. The class of partially ordered spaces $\mathscr{E}$ that are of Dedekind type is quite broad. It contains e.g. the following "classical" Banach spaces: R^n; l^p, $p \geqslant 1$; $L^p(\Omega)$, $1 \leqslant p < +\infty$, where Ω is a subset of R^n—if their partial order is implied by the standard cones $\mathscr{R}^n_+$, l^p_+, and $\mathscr{L}^p_+$ respectively consisting of vectors or functions assuming only nonnegative values (in an appropriate sense). Although the space $\mathscr{C}([0,1])$ ordered by the standard cone $\mathscr{C}_+([0,1])$ is a Banach lattice, it does not possess the monotone convergence property e.g. $x_k(t) = t^k$, $k = 1, 2, \dots \Rightarrow x_k \leqslant x_{k+1} \leqslant e$, where $e(t) \equiv 1$.

It is easy to see that every finite dimensional space partially ordered by a solid cone is of Dedekind type. This is because every cone nondecreasing bounded sequence is weakly convergent and thus, in finite dimensional spaces, norm convergent. Since among finite dimensional spaces there are obviously such spaces ordered by cones which are not lattice type cones, there exist Dedekind type spaces which are not Banach lattices.

An operator $T \in B(\mathscr{E})$ is called *$\mathscr{K}$-nonnegative* [7] if $T\mathscr{K} \subset \mathscr{K}$. A $\mathscr{K}$-nonnegative operator T is called *$\mathscr{K}$-irreducible* [10] if for every pair $x \in \mathscr{K}$, $x \neq 0$, $x' \in \mathscr{K}'$, $x' \neq 0$, there is an index $p = p(x, x') \geqslant 1$, such that $[T^p x, x'] > 0$.

Let

$$\mathscr{V}_{\mathscr{K}} = \{ V \in B(\mathscr{E}) : V\mathscr{K} \subset \mathscr{K} \}.$$

It is well known [11] that $\mathscr{V}_{\mathscr{K}}$ is a normal closed not necessarily generating cone in the Banach space $B\mathscr{E}$, i.e. properties (i)–(iv) and (vi) in the definition of a cone hold. It is easy to see that if $\mathscr{E}$ is of Dedekind type with the order given by $\mathscr{K}$, then $B(\mathscr{E})$ is of Dedekind type with the order implied by $\mathscr{V}_{\mathscr{K}}$.

Let $S, T \in B(\mathscr{E})$. We let

$$T \geqslant S \ \text{(or equivalently } S \leqslant T) \quad \Leftrightarrow \quad (T - S)\mathscr{K} \subset \mathscr{K}.$$

Let $T \in B(\mathscr{E})$. By $\tilde{T}$ we denote the *complex extension* of T, i.e. $\tilde{T}z = Tx + iTy$, where $z = x + iy$, $x, y \in \mathscr{E}$.

Let I denote the identity operator. Let $T \in B(\mathscr{E})$, and $\tilde{T}$ be its complex extension. The set

$$\rho(T) = \left\{\lambda \in \mathscr{C} : (\mathrm{I} - \tilde{T})^{-1} \in B(\mathscr{C})\right\}$$

is called the *resolvent set* of T. Its complement

$$\sigma(\tilde{T}) = \mathscr{C} \setminus \rho(\tilde{T})$$

is called the *spectrum* of T. By definition, $\sigma(T) \equiv \sigma(\tilde{T})$.

The quantity

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$$

is called the *spectral radius* of T.

We define the *peripheral spectrum* of T by setting

$$\pi\sigma(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}.$$

An operator $A \in B(\mathscr{E})$ is called an *M-operator* (more precisely a $\mathscr{K}$-*M-operator* [9]) if $A = bI - B$, where $b \geq r(B)$ and $B\mathscr{K} \subset \mathscr{K}$.

Let $\tilde{T} \in B(\mathscr{F})$, and let $\mu \in \sigma(\tilde{T})$ be isolated. Then [12, pp. 328–329]

$$(\lambda I - \tilde{T})^{-1} = \sum_{k=0}^{\infty} A_k(\mu)(\lambda - \mu)^k + \sum_{k=1}^{\infty} B_k(\mu)(\lambda - \mu)^{-k}, \quad (1)$$

where the $A_k(\mu)$, $B_{k+1}(\mu) \in B(\mathscr{F})$ and the following relations hold:

$$[B_1(\mu)]^2 = B_1(\mu) \quad (2)$$

and

$$B_{k+1}(\mu) = (\tilde{T} - \mu I)B_k(\mu)k = 0, 1, \ldots . \quad (3)$$

In particular, if there is an index $q = q(\mu) < +\infty$ such that $B_k(\mu) = 0$ for $k > q$, the singularity μ is called a *pole* of the resolvent operator $(\lambda I - \tilde{T})^{-1}$ and $q = q(\mu)$ is called its *multiplicity* or *order*. We write $q(\mu) = 0$ if $\mu \notin \sigma(T)$, i.e. if $\mu \in \rho(T)$ is a *regular point* of the resolvent operator.

An operator $\tilde{T} \in B(\mathscr{F})$ is said to have *property "p"* if the peripheral spectrum $\pi\sigma(\tilde{T})$ consists of poles of the resolvent operator. We say that

$T \in B(\mathscr{E})$ has property "p" if its complex extension possesses this property. Similarly we say that an operator $T \in B(\mathscr{F})$ has a certain property if its complex extension $\tilde{T}$ possesses this property.

An operator $\tilde{T} \in B(\mathscr{F})$ is called *convergent* if there exists an operator $P \in B(\mathscr{F})$ such that $0 = \lim_{k \to \infty} \|\tilde{T}^k - P\|_{\mathscr{F}}$. If $P = 0$ (the zero operator), $\tilde{T}$ is called *zero convergent*.

The following two propositions are slight generalizations of the results stated in [1]; see also [2, p. 152].

PROPOSITION 1. *Let $\tilde{T} \in B(\mathscr{F})$ have property "p." Then $\tilde{T}$ is convergent if and only if $\tilde{T} = P + S$, where $P^2 = P$, $PS = SP = 0$, and $r(S) < 1$.*

PROPOSITION 2. *Let $T \in B(\mathscr{E})$ and let $T\mathscr{K} \subset \mathscr{K}$. Moreover, let $T \geqslant dI$, $d > 0$, and the value $r(T)$ be a pole of the resolvent operator $(\lambda I - T)^{-1}$.*

Then $r(T) \geqslant d$ and $[1/r(T)]T = V$, $r(V) = 1$, is convergent if and only if $q = 1$, where q is the multiplicity of 1 as a pole of the resolvent operator $(\lambda I - V)^{-1}$.

Proof. Since $T \geqslant dI$, we have $r(T) \geqslant d$ by the Frobenius comparison theorem [8], and therefore $r(V) = 1$.

Let V be convergent, and let $P = \lim_{k \to \infty} V^k$. It follows that $(V - I)P = 0$ and thus, by (3) with P in place of $B_1(\mu)$, that $q = q(1) = 1$.

Conversely, if $q = q(1) = 1$, then $V = B_1(1) + S$ (see [12, Theorem 10.1, p. 330]), where $[B_1(1)]^2 = B_1(1)$, $B_1(1)S = SB_1(1) = 0$, and $r(S) \leqslant 1$. Let $\lambda \in \sigma(V)$, $|\lambda| = 1$. Since

$$V \geqslant \frac{d}{r(T)} I,$$

we can write $r(T)V = dI + R$, where $R\mathscr{K} \subset \mathscr{K}$. It follows that $r(R) \leqslant r(T) - d < r(T)$ and therefore

$$[r(T)]^2 = (d + \mu)^2 + \nu^2 \leqslant d^2 + 2d\mu + [r(R)]^2,$$

where $\mu = \mathscr{R}\lambda$, $\nu = \mathscr{I}\lambda$ for some $\lambda \in \sigma(T)$. If $\nu \neq 0$, then $\mu < r(T) - d$ and the above relations are contradictory. Thus, $\nu = 0$. The case $\lambda < 0$, $|\lambda| = r(V)$, is contradictory too, because then $\lambda = d + \mu$, $\mu < 0$, and $|\mu| = |\lambda| + d > r(T)$. ∎

An M-operator $A \in B(\mathscr{E})$ is said to have *property "c"* if there is an operator $B \in B(\mathscr{E})$ such that $b \geqslant r(B)$, $A = bI - B$, and $(1/b)B = T$ is convergent. It is easy to see that each M-operator for which $b > r(B)$ has property "c." On the other hand, it is known that even in the matrix case there are singular M-matrices not having property "c."

Let us introduce the class of operators $\mathscr{Z}(\mathscr{K})$ by setting

$$\mathscr{Z}(\mathscr{K}) = \{ A \in B(\mathscr{E}) : A = bI - B,\ B\mathscr{K} \subset \mathscr{K},\ b \text{ real}\}.$$

As in the classical case, the $\mathscr{K} - M$-operators possess the following well-known properties.

THEOREM 1. *Let $B \in B(\mathscr{E})$ have property "p."*

(*a*) *An operator $A = bI - B$, $A \in \mathscr{Z}(\mathscr{K})$, is an M-operator with $b > r(B)$ if and only if $A^{-1} \in B(\mathscr{E})$ and $A^{-1}\mathscr{K} \subset \mathscr{K}$.*

(b) *$A \in \mathscr{Z}(\mathscr{K})$ is an M-operator with $b \geqslant r(B)$ if and only if*

$$\sigma(A) \subset \{\lambda : \mathscr{R}\lambda > 0\} \cup \{0\}. \tag{4}$$

(c) *If A is an M-operator having property "c" and $A^2 \in \mathscr{Z}(\mathscr{K})$, then A^2 is an M-operator having property "c" too.*

Proof. (a): Let $A = bI - B$ be such that $A^{-1} \in B(\mathscr{E})$ and $A^{-1}\mathscr{K} \subset \mathscr{K}$. It follows that $b \neq r(B)$.

Let $b < r(B)$. Let $x_0 \in \mathscr{K}$, $x_0 \neq 0$, be such that $Bx_0 = r(Bx_0$ (see [8]). We see that

$$0 \leqslant A^{-1}x_0 = (bI - B)^{-1}x_0 = \frac{1}{b - r(B)}x_0 \leqslant 0,$$

a contradiction.

On the other hand, if $b > r(T)$, then

$$A^{-1} = (bI - B)^{-1} = \frac{1}{b}\left(I - \frac{1}{b}B\right)^{-1} = \frac{1}{b}\sum_{k=0}^{\infty}\left(\frac{1}{b}B\right)^k.$$

Thus, (a) is proved.

(b): As a consequence of the definition of the spectrum, $\lambda \in \sigma(A)$ can be expressed as $\lambda = b - \beta$, $\beta \in \sigma(B)$.

Let A be an M-operator. From $|\beta| \leqslant r(B)$ we deduce that $\mathscr{R}\lambda = b - \mathscr{R}\beta \geqslant 0$. It follows that for $b \geqslant r(B)$, $\mathscr{R}\lambda \geqslant 0$. Moreover, $\min\{\mathscr{R}\lambda : \lambda \in$

$\sigma(B)\} > 0$ if and only if $b > r(B)$. We show that if $b = r(B)$ and $\mathscr{R}\lambda = 0$, then $\lambda = 0$.

Let $i\zeta \in \sigma(A)$ for some $0 \neq \zeta$ real. Then $i\zeta = b - \beta$ that is $\beta = b - i\zeta$. However, $|\beta|^2 = b^2 + \zeta^2 > r(B)$, a contradiction. Thus, the necessity of (4) follows.

Since the sufficiency of the condition (4) is obvious, assertion (b) is proved.

(c): According to [9] it is enough to show that the relations $A^2x \in \mathscr{K}$, $x = A^2y$, imply that $x \in \mathscr{K}$. However, if we let $u = Ax$ and $v = Au$, we see that $v \in \mathscr{K}$ and therefore, since A is an M-operator, $u \in \mathscr{K}$. By the same argument, since $Ax = u$ and $x = A^2y$, we deduce that $x \in \mathscr{K}$, and this completes the proof of (c) as well as of Theorem 1. ■

3. SQUARE ROOTS OF M-OPERATORS

Let A be an M-operator, and let

$$A = bI - B, \qquad b \geqslant r(B), \quad B\mathscr{K} \subset \mathscr{K}.$$

It is easy to see that b can be chosen positive; let $b > 0$.

Since for every $c > 0$

$$\frac{1}{b}A = (1 + c)I - \left(cI + \frac{1}{b}B\right),$$

we see that

$$\frac{1}{b(1 + c)}A = I - T,$$

where

$$T = \frac{1}{b(1 + c)}\left(cI + \frac{1}{b}B\right), \qquad r(T) \leqslant 1, \quad T \geqslant dI,$$

with $d = c/b(1 + c) > 0$. Therefore, we can assume in the sequel that $A = t(I - T)$, where $t > 0$ and $T\mathscr{K} \subset \mathscr{K}$, $T \geqslant dI$, $d > 0$, $r(T) \leqslant 1$.

It is obvious that A has an M-operator as a square root if and only if $I - T$ has an M-operator as a square root.

As in the classical situation, we have the following result.

LEMMA 1. *Let $\mathscr{E} = \mathscr{K} - \mathscr{K}$ be a Dedekind type space. Let $T \in B(\mathscr{E})$ have property "p." Furthermore, let $r(T) \leqslant 1$, $T \geqslant dI$, $d > 0$. Then:*

(a) *There exists an operator $B \in B(\mathscr{E}E)$, $B\mathscr{K} \subset \mathscr{K}$, such that $r(B) \leqslant \alpha$ and $I - T = (\alpha I - B)^2$ if and only if the iterative process*

$$X_{k+1} = \frac{1}{2\alpha}\{T + (\alpha^2 - 1)I + X_k^2\}, \qquad X_0 = 0, \tag{5}$$

is norm convergent. In that case

$$B \geqslant X = \lim_{k \to \infty} X_k, \qquad X\mathscr{K} \subset \mathscr{K}, \quad r(X) \leqslant \alpha,$$

and there is a real $c > 0$ such that

$$X \geqslant cI \quad \text{and} \quad (\alpha I - X)^2 = I - T.$$

(b) *If the process (5) is norm convergent, then T and $(1/\alpha)X$ are convergent.*

(c) *If T is convergent, then (5) is norm convergent for all $\alpha \geqslant 1$. Denoting in this case the limit of the iteration process*

$$Y_{k+1} = \tfrac{1}{2}\left[T + Y_k^2\right], \qquad Y_0 = 0, \tag{6}$$

by Y, the relation

$$\alpha I - X = I - Y \tag{7}$$

holds.

In the proof of Lemma 1 some auxiliary results are required; we present them in Lemmas 2–4.

LEMMA 2. *Let $T \in B(\mathscr{E})$, $T\mathscr{K} \subset \mathscr{K}$, have property "p," and let f be a polynomial with nonnegative coefficients. Then $f(T)\mathscr{K} \subset \mathscr{K}$, and*

$$r(f(T)) \leqslant f(r(T)).$$

Proof. The $\mathscr{K}$-nonnegativity of $f(T)$ is obvious.

By the spectral mapping theorem [12, Theorem 9.5, p. 325],

$$\sigma(f(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}$$

and thus

$$r(f(T)) = \max\{|f(\lambda)| : \lambda \in \sigma(T)\}.$$

Since by the Schaefer's theorem [11]

$$0 \leqslant r(f(T)) \in \sigma(T),$$

we see that

$$r(f(T)) = f(\hat{\lambda}), \qquad \hat{\lambda} \in \sigma(T).$$

Because of the relation

$$|f(|\lambda|)| \leqslant f(r(T)) \qquad \text{for} \quad |\lambda| \leqslant r(T),$$

the validity of the statement follows. ■

LEMMA 3. *Let $B_{j,1}$ be the spectral projection corresponding to an isolated point $\lambda_j \in \sigma(T)$ (analogous to $B_1(\mu)$ in (1) for $\mu = \lambda_j$).*

$$X_k B_{j,1} = \sum_{t=1}^{g_j} a_{jt}^{(k)} B_{j,t},$$

where $\alpha_{jt}^{(k)}$ are some (complex) numbers.

Proof. This is an easy consequence of the definition of the sequence $\{X_k\}$ and the formulas (2), (3). ■

LEMMA 4. *Let Ω and Ω' be bounded open sets in the complex plane. Let the functions ϕ and ψ be holomorphic on Ω and Ω', respectively, where*

$$\sigma(C) \subset \Omega \quad \text{and} \quad \{\mu : \mu = \phi(\lambda), \lambda \in \sigma(C)\} \subset \Omega'.$$

Let $C \in B(\mathscr{F})$ and $S = \phi(C)$.
Then

$$\chi(C) = \psi(S),$$

where

$$\chi(\lambda) = \psi[\phi(\lambda)] \qquad \textit{for} \quad \lambda \in \Omega.$$

REMARK 2. This result can be found in [3, VII.3.12, p. 570]. A similar result for unbounded operators is in [12, Theorem 9.7, p. 326].

COROLLARY 1. *Let $T \in \mathscr{E}$, $T\mathscr{K} \subset \mathscr{K}$, $r(T) \leqslant 1$ be convergent. Let $F = \gamma I - H$ be an M-operator that belongs to the operator algebra generated by T and its spectral resolution. Let*

$$F^2 = I - T. \tag{8}$$

Then

$$F = \psi(I - T), \tag{9}$$

where

$$\psi(\mu) = \sqrt{\mu}, \qquad \mu \in \Omega' = \{\mu: \mathscr{R}\mu > 0\}.$$

Proof. First, assume that $r(T) = 1$. Let us write T in the form

$$T = P + T_0,$$

where P is the Perron eigenprojection. The relation (8) implies that

$$F^2P = (I - T)P = 0,$$

and thus, since $I - T$ possesses property "c,"

$$FP = 0,$$

showing that F possesses property "c" too.
Further, from

$$F^2 = F^2(I - P) = I - T = (I - T_0)(I - P),$$

we deduce that

$$(I - T_0)(I - P) = \phi(F(I - P)),$$

where

$$\phi(\lambda) = \lambda^2.$$

Let

$$C = F(I - P)$$

and

$$\Omega = \{\lambda \in \mathscr{C}^1 : \mathscr{R}\lambda > 0\}.$$

Since both $I - T$ and C are M-operators, we see that

$$\sigma((I - T)(I - P)) \cup \sigma(C(I - P)) \subset \Omega.$$

Since $C^2 = I - T$, we deduce that $\sigma(C^2) \subset \Omega$; thus,

$$\lambda \in \sigma(C) \quad \Rightarrow \quad \phi(\lambda) = \mu \in \sigma((I - T)(I - P)) \subset \Omega.$$

We see that Lemma 4 applies with $\phi(\lambda) = \lambda^2$ and $\psi(\mu) = \mu^{1/2}$. Therefore,

$$\begin{aligned} C = \chi(C) = F(I - P) &= \psi((I - T_0)(I - P)) \\ &= \psi((I - T)(I - P)) = \psi(I - T). \end{aligned}$$

Corollary 1 is thus proven for the case $r(T) = 1$.

Second, assume that $r(T) < 1$. Then $0 \notin \sigma(I - T) \cap \sigma(F) \cap \sigma(F^2)$. Set $C = F$, $S = \phi(F)$ with ϕ and ψ as before. Again, Lemma 4 applies, and thus (9) holds for this case too. The proof is complete. ■

The following example shows that the hypothesis in Corollary that a square root of $I - T$ belongs to the operator algebra generated by T and its spectral resolution is essential.

EXAMPLE 1. Let $T \in B(\mathscr{E})$, $\mathscr{T}\mathscr{K} \subset \mathscr{K}$, $r(T) = 1$, $T = P + T_0$, where P denotes the Perron eigenprojection, have property "p." Let dim $\mathscr{R}(P) = N$, where $\mathscr{R}(P)$ denotes the range of P. Let us choose the standard basis in

$\mathscr{R}(P)$. Define operators $Q_{jk} = PN_{jk} = N_{jk}P$ by setting

$$N_{jk} = \left(n_{tl}^{(j,k)}\right),$$

where

$$n_{tl}^{(j,k)} = \begin{cases} -b & \text{for} \quad t = j,\ l = k,\ j \neq k, \\ 0 & \text{otherwise,} \end{cases}$$

where b is a positive real. Let $I - Y$ be the square root of $I - T$ constructed by the iterative process (6).

Defining

$$C_{jk} = Q_{jk} + I - Y = Q_{jk}P + (I - Y)(I - P),$$

we check easily that

$$C_{jk}^2 = (I - Y)^2 = I - T.$$

Thus, the number of linearly independent square roots of $I - T$ is bounded below by $N(N - 1) + 1$. Consequently, there exist an infinite number of linearly independent square roots of $I - T$ if $\dim \mathscr{R}(P) = +\infty$.

REMARK 3. It is easy to see that the square roots constructed in Example 1 do not belong to the operator algebra generated by T and its spectral resolution. Indeed, if restricting to the subalgebra generated by T and P, we get just the operators of the form ρP with some scalar ρ. Obviously, this is not the case for Q_{jk}.

Proof of Lemma 1. (a) $\Rightarrow$: We assume that there exists a $B \in B(\mathscr{E})$, $B\mathscr{K} \subset \mathscr{K}$, such that $r(B) \leqslant \alpha$ and $I - T = (\alpha I - B)^2$. We prove by induction that the sequence which is computed using (5) is $\mathscr{K}$-nondecreasing and bounded. Assuming $X_{k+1} \geqslant X_k$, $X_k \leqslant B$, which is true for $k = 0$, and using the fact that $0 \leqslant X_k \leqslant B$ implies $X_k^2 \leqslant B^2$, we deduce that

$$X_{k+2} = \frac{1}{2\alpha}\left[T + (\alpha^2 - 1)I + X_{k+1}^2\right]$$

$$\geqslant \frac{1}{2\alpha}\left[T + (\alpha^2 - 1)I + X_k^2\right] = X_{k+1},$$

$$B - X_{k+1} = B - \frac{1}{2\alpha}\left[T + (\alpha^2 - 1)I + X_k^2\right]$$

$$= \frac{1}{2\alpha}\left[2\alpha B - T - \alpha^2 I + I - X_k^2\right]$$

$$\geqslant \frac{1}{2\alpha}\left[I - T - (\alpha^2 I - 2\alpha B + B^2)\right] = 0.$$

Since $\mathscr{E}$ is a Dedekind type space, we deduce that the sequence $\{X_k\}$ is norm convergent. Thus, there exists the norm limit $X = \lim_{k\to\infty} X_k$. It follows that

$$X \leqslant B, \qquad X \geqslant dI, \qquad (\alpha I - X)^2 = I - T,$$

and consequently, by [8],

$$r(X) \leqslant r(B) \leqslant \alpha.$$

$\Leftarrow$: Let $X = \lim_{k\to\infty} X_k$. Then

$$I - T = (\alpha I - X)^2.$$

We show that $r(X) \leqslant \alpha$. To this purpose let us notice that X_k is a polynomial in T with nonnegative coefficients. By Lemma 2

$$r(X_{k+1}) \leqslant r\left(\frac{1}{2\alpha}\left[T + (\alpha^2 - 1) + X_k^2\right]\right)$$

$$\leqslant \frac{1}{2\alpha}\left[r(T) + \alpha^2 - 1 + r\left(X_k^2\right)\right] \leqslant \alpha.$$

Therefore, assertion (a) holds with $B = X$.

(b): Let the process (5) be convergent, and assume that T is not convergent. Since $T \geqslant dI$, $d > 0$, Proposition 2 implies that $q = q_1 > 1$, where $q_1 = q(1)$ is the multiplicity of 1 as a pole of the resolvent operator $(\lambda I - T)^{-1}$. By Lemma 2, the relations

$$X_k B_{1,1} = \sum_{t=1}^{q} a_{1t}^{(k)} B_{1,t}$$

and the norm convergence of $\{X_k\}$ imply that

$$XB_{1,1} = \sum_{t=1}^{q} a_{1t} B_{1,t}.$$

Therefore, on the one hand,

$$(\alpha B_{1,1} - XB_{1,1})^2 B_{1,1} = \left(\alpha B_{1,1} - \sum_{t=1}^{q} a_{1t} B_{1,t}\right)^2,$$

and on the other hand,

$$(I - T) B_{1,1} = -B_{1,2}.$$

The equality $(\alpha I - X)^2 = I - T$ implies that $a_{11} = \alpha$ and then necessarily $B_{1,2} = 0$, otherwise a contradiction. Thus, T is convergent.

In case $\alpha > r(X)$, $(1/\alpha)X$ is zero convergent.

Let $\alpha = r(X)$. Let $T = P + S$, $PS = SP = 0$, $r(S) < 1$, with $P = B_{1,1}$, the eigenprojection of T corresponding to eigenvalue 1. Then with the appropriate polynomials f_k,

$$\begin{aligned}\frac{1}{\alpha} X &= \lim_{k \to \infty} f_k(T)\\ &= \lim_{k \to \infty} f_k(P) + \lim_{k \to \infty} f_k(S)\\ &= \lim_{k \to \infty} f_k(1) P + \lim_{k \to \infty} f_k(S)(I - P).\end{aligned}$$

From this expression it follows that value $\lim_{k \to \infty} f_k(1) = (1/\alpha) r(X)$ is a simple pole of the resolvent $[\lambda I - (1/\alpha)X]^{-1}$. Moreover, since $(1/\alpha)X \geqslant$

$(d/2\alpha)I$, according to Proposition 2, $(1/\alpha)r(X)$ is the only element of the peripheral spectrum $\pi\sigma(T)$. It follows that $\alpha I - X$ is an M-operator having property "c."

The hypothesis that the cone $\mathscr{K}$ is normal implies [6, Theorem 4.2, p. 52] the existence of a norm $\|\cdot\|_m$ equivalent with $\|\cdot\|_{\mathscr{E}}$ which is monotone, i.e.

$$\|x + y\|_m \geqslant \|x\|_m, \qquad x, y \in \mathscr{K}.$$

Let us define a new norm $\|\cdot\|_T$ by setting

$$\|x\|_T = \sum_{k=0}^{\infty} \left\|S^k(I - P)x\right\|_m + \|Px\|_m.$$

We see that this norm is equivalent to the m-norm $\|\cdot\|_m$ as well as to the original norm $\|\cdot\|_{\mathscr{E}}$, because

$$\|S^k\|_m \leqslant \kappa(1 - \eta)^k \qquad \text{for some } \eta > 0,\ \kappa > 0 \text{ independent of } k,$$

and

$$\|x\|_m \leqslant \|x\|_T \leqslant \left(\|P\|_m + \sum_{k=0}^{\infty} \|S^k\|_m \|(I - P)\|_m\right)\|x\|_m$$

$$\leqslant \left(\|P\|_m + \frac{\kappa}{\eta}\|I - P\|_m\right)\|x\|_m,$$

and the fact that the norms $\|\cdot\|_m$ and $\|\cdot\|_{\mathscr{E}}$ are equivalent.

Moreover,

$$\|T\|_T = \sup\left(\frac{\|Px\|_m + \sum_{k=0}^{\infty}\left\|S^{k+1}(I - P)x\right\|_m}{\|Px\|_m + \sum_{k=0}^{\infty}\left\|S^k(I - P)x\right\|_m} : x \neq 0\right) \leqslant 1. \tag{10}$$

By induction we show that $\|X_k\|_T \leqslant \alpha$, $k = 0, 1, 2, \ldots$ For $k = 0$ it is trivial, because $X_0 = 0$. Let $\|X_k\|_T \leqslant \alpha$. Then, using (10),

$$\|X_{k+1}\|_T \leqslant \frac{1}{2\alpha}\left(\|T\|_T + \alpha^2 - 1 + \|X_k\|_T^2\right)$$

$$\leqslant \frac{1}{2\alpha}(1 + \alpha^2 - 1 + \alpha^2) = \alpha. \tag{11}$$

Hence, the relations $X_k \leqslant X_{k+1}$, $k = 0, 1, \dots,$ together with (11) imply that

$$\|X_k\|_T \leqslant \|X_{k+1}\|_T \leqslant \alpha, \qquad k = 0, 1, \dots .$$

It follows that the sequence $\{X_k\}$ is norm convergent.

Let Y_k be defined by (6), and let $Y = \lim_{k\to\infty} Y_k$. To show the validity of (7) we check that Corollary 1 applies with T considered and $A = \alpha I - X$ or $B = I - Y$ in place of C.

By construction, the operators A and B belong to the operator algebra generated by T and its spectral resolution.

We see that Corollary 1 applies and consequently,

$$\alpha I - X = A = \psi(I - T) = B = I - Y.$$

This completes the proof of Lemma 1. ∎

The following result is a full analogue of Theorem 4 in [1].

THEOREM 2. *Let $\mathscr{K}$ be a generating closed normal cone in $\mathscr{E}$, and let $\mathscr{E}$ ordered by $\mathscr{K}$ be a Dedekind type space. Let $A \in B(\mathscr{E})$ be an M-operator, and let $A = t(I - T)$ be its representation such that $T\mathrm{K} \subset \mathscr{K}$, $r(T) = 1$, $T \geqslant dI$, $d > 0$. Then A has an M-operator as a square root if and only if A has property "c." In this case let Y denote the limit of the process generated by*

$$Y_{k+1} = \tfrac{1}{2}\left(T + Y_k^2\right), \qquad Y_0 = 0.$$

The map $\sqrt{t}\,(I - Y)$ is an M-operator with property "c," and $t(I - Y)^2 = A$. Moreover, for any M-operator Z which is a square root of A the relation $Z \leqslant \sqrt{t}\,(I - Y)$ holds.

Proof. The proof is very similar to that of the M-matrix case.

If A has an M-operator as a square root, then we see from Lemma 1, parts (a) and (b), that T is convergent and hence A has property "c." The other direction follows from Lemma 1, part (b); we also obtain that the operator $\sqrt{t}(I - Y)$ has property "c."

Finally, let Z be an M-operator which is a square root of A, and let $Z = \beta I - B$, $\beta \geqslant \sqrt{t}$, be a representation in which $r(B) = \beta$, $B \geqslant bI$, $b > 0$. The operator $I - T$ can be expressed as

$$I - T = \frac{1}{t}A = \left(\frac{\beta v}{\sqrt{t}}I - \frac{1}{\sqrt{t}}B\right)^2,$$

where $\beta \geqslant \sqrt{t} \geqslant 1$. Setting $\alpha = \beta/\sqrt{t}$ and denoting by X the norm limit of the sequence generated by

$$X_{k+1} = \frac{1}{2\alpha}\left[T + (\alpha^2 - 1) + X_k^2\right], \qquad X_0 = 0,$$

we get from Lemma 1, parts (a) and (c), that $X \leqslant (1/\sqrt{t})B$ and $\alpha I - (1/\sqrt{t})B \leqslant \alpha I - X = I - Y$, that is,

$$\sqrt{t}(I - Y) \geqslant \beta I - B = Z.$$

This shows that the operator $\sqrt{t}(I - Y)$ is the largest square root of A. The proof of Theorem 2 is thus complete. ∎

The next result is also analogous to that presented in [1]. We should note that the class of M-operators possessing the property that each of its elements has a unique M-operator as a square root contains as subclass the class of nonsingular M-operators and the subclass of singular M-operators whose generalized eigenspaces are one dimensional.

THEOREM 3. *Let $\mathscr{K}$ be a generating closed normal cone in $\mathscr{E}$, and let $\mathscr{E}$ ordered by $\mathscr{K}$ be a Dedekind type space. Let $A = t(I - T)$, $r(T) = 1$, $T \geqslant dI$, $d > 0$, and let T have property "p." An M-operator A has exactly one M-operator as a square root if the generalized eigenspace of A corresponding to the eigenvalue* 0, *or equivalently, the generalized eigenspace of T corresponding to the eigenvalue* 1, *is one dimensional.*

Proof. By our hypothesis,

$$\bigcup_{k=1}^{\infty} \ker A^k = \ker A$$

and thus A has property "c." Theorem 4 guarantees the existence of a square root of A and in particular that $I - Y$, where $Y = \lim_{k \to \infty} Y_k$ defined by (6), is an M-operator which is a square root of $t(I - T)$. We assume that there exists another M-operator Z which is a square root of $t(I - T)$ and that $Z = \alpha I - B$, $\alpha \geqslant 1$, $r(B) = \alpha$, $B \geqslant bI$, $b > 0$.

let us assume that

$$B = r(B)Q + [B - r(B)I]Q + R,$$

where

$$Q^2 = Q, \qquad QR = RQ = 0, \qquad r(B) \notin \sigma(R), \qquad [B - r(B)I]Q \neq 0.$$

Let $y = [B - r(B)I]Qx$. Then $[B - r(B)I]y = 0 = [B - r(B)I]^2 Qx$ and hence $t(I - T)x = t(I - T)y = 0$. This implies that x and y are linearly dependent, a contradiction. Therefore, $[B - r(B)I]Q = 0$; thus Z has the property "c."

To show the equality of Z and $I - Y$ we check that

$$Z^2(I - Q) = Z^2 = t(I - T) = t(I - T)(I - P).$$

Therefore, either $Z = 0 = t(I - T) = I - Y$, or $Q = P$. It follows that in either case, Z belongs to the operator algebra generated by T and its spectral resolution. According to Corollary 1, $Z = \psi(I - T) = I - Y$, and this completes the proof of Theorem 3. ■

4. SOME CHARACTERIZATIONS OF M-OPERATORS AND THEIR SQUARE ROOTS

In this section we present some characterizations of M-operators in the spirit of [1]. The corresponding proofs are verbally the same and are thus omitted.

THEOREM 4. *Let $A = t(I - T)$, where $T \in B(\mathscr{E})$, $T\mathscr{K} \subset \mathscr{K}$, $r(T) < 1$, have property "p." Then A is an M-operator if and only if there exists an M-operator Y, $0 \notin \sigma(Y)$, such that $A = Y^2$.*

THEOREM 5. *Let $A = t(I - T)$, where $T \in B(\mathscr{E})$, $T\mathscr{K} \subset \mathscr{K}$, $r(T) < 1$, have property $= =$ p." Then A is an M-operator if and only if there exists an operator $Z \in B(\mathscr{E})$, $Z\mathscr{K} \subset \mathscr{K}$, such that $AZ^2 = I$.*

THEOREM 6. *Let $A = t(I - T)$, where $T \in B(\mathscr{E})$, $T\mathscr{K} \subset \mathscr{K}$, $r(T) = 1$, have property "p." Furthermore, let $r(T)$ be a simple pole of the resolvent operator $(\lambda I - T)^{-1}$. Then A is an M-operator which has property "c" if and only if there exists an M-operator $Z \in B(\mathscr{E})$ such that $A = Z^2$.*

5. CONCLUDING REMARKS

The theory developed in this contribution can be generalized further. In particular, higher order roots can be considered. A fairly general study of such possible extensions is presented in [4] for the case of some classes of analytic functions such as *totally nonnegative* functions, *totally oscillating* functions, and functions forming *reciprocating pairs*. In this respect the results obtained in [4] are more general than our results. On the other hand, our theory admits more general cones and therefore more general classes of M-operators. The most important difference is that the authors of [4] investigate regular M-matrices. As we show, the presence of 0 in the spectrum of the M-operator under consideration implies possible nonuniqueness result which do not occur in the case of square roots of regular M-matrices.

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REFERENCES

1 G. Alefeld and N. Schneider, On square roots of M-matrices, *Linear Algebra Appl.* 42:119–132 (1982).

2 A. Berman and R. Plemmons, *Non-negative Matrices in the Mathematical Sciences*, Academic, 1979.

3 N. Dunford and J. T. Schwartz, *Operator Theory. Part I. General Theory*, Interscience, New York, 1958.

4 M. Fiedler and H. Schneider, Analytic functions of M-matrices and generalizations, *Linear and Multilinear Algebra* 13:185–201 (1983).

5 F. R. Gantmacher, *The Theory of Matrices* (in Russian), Gos. Izd. Tech. Lit., Moscow, 1954.

6 M. A. Krasnoselskii, Je. A. Lifschits, and A. V. Sobolev, *Positive Linear Systems* (in Russian), Nauka, Moscow, 1985.

7 M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space (in Russian), *Uspekhi Mat. Nauk* III(1):3–95 (1948); English transl., *Amer. Math. Soc. Transl.* 26 (1950).

8 I. Marek, Frobenius theory of positive operators. Comparison theorems and applications, *SIAM J. Appl. Math.* 19:607–628 (1970).

9 I. Marek and D. Szyld, Splittings of M-operators: Irreducibility and the index of the iteration operator, *Numer. Funct. Anal. Optim.* 11:529–553 (1990).

10 I. Sawashima, On spectral properties of some positive operators, *Rep. Nat. Sci. Ochanomizu Univ.* 15:53–64 (1964).

11 H. H. Schaefer,On the singularities of an analytic function with values in Banach space, *Arch. Math. (Basel)* XI:40–43 (1960).

12 A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., Wiley, New York, 1980.

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